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STEADY MOTIONS OF A RIGID BODY IN A CENTRAL GRAVITATIONAL FIELD†

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The problem of the two-dimensional translational–rotational motion of a rigid body in a central gravitational field is considered. All steady motions are found, their stability is investigated, and bifurcation diagrams are constructed. New effects resulting from the use of the exact expression for the gravitational potential are discovered.

1. The problem of the motion of an oblate body in a central gravitational field is considered. We model the body by a massless disk of radius a, with point masses $m_i/2$ $(i=1, 2)$ attached at the opposite ends of two mutually perpendicular diameters d_i.

We assume that the disk moves in a plane containing the centre of attraction. Then the position of the disk is completely specified by three generalized coordinates: the distance r from the centre of mass C of the body to the centre of attraction O, the angle θ between the line OC and the diameter d_1, and the angle φ between some fixed direction in the plane of the motion and the OC axis. The kinetic energy T and potential energy V of the disk take the form

$$T = \tfrac{1}{2} m[\dot{r}^2 + r^2\dot{\varphi}^2 + a^2(\dot{\theta}+\dot{\varphi})^2]$$
$$V = F_1(a) + F_1(-a) + F_2(a) + F_2(-a)$$
$$F_i(a) = -\tfrac{1}{2} fMm_i(r^2 + 2ra\gamma_i + a^2)^{-1/2} \quad (i=1,2)$$
$$\gamma_1 = \cos\theta, \quad \gamma_2 = \sin\theta$$

Here M is the mass of the centre of attraction, $m = m_1 + m_2$ is the mass of the body, and f is the gravitational constant.

The Lagrangian $T-V$ does not depend on the angle φ. Hence the equations of motion admit of the area integral

$$\partial T / \partial\dot{\varphi} = k = \text{const} \tag{1.1}$$

in addition to the energy integral $T+V=h=\text{const}$ and the body can perform motions of the form

$$r = \text{const}, \quad \theta = \text{const}, \quad \dot{\varphi} = \text{const} \tag{1.2}$$

Here the centre of mass of the body rotates uniformly about the centre of attraction, and the body preserves a constant orientation with respect to that centre.

Ignoring the cyclic variable φ, we introduce the Routhian $R+R(\dot{r}, \theta, r, \theta, k)$ by the relation

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$R=T-V-k\dot{\varphi}$, in which the variable $\dot{\varphi}$ has been eliminated from the right-hand side by using the area integral (1.1). The Routhian has the form

$$R=\frac{1}{2}m\left(\dot{r}^2+\frac{r^2a^2}{r^2+a^2}\dot{\theta}^2\right)+k\frac{a^2}{r^2+a^2}\dot{\theta}-V-\frac{1}{2}\frac{k^2}{m(r^2+a^2)}\equiv R_2+R_1+R_0$$

where R_s is a homogeneous form of the positional velocities $\dot{r}$ and $\dot{\theta}$ of degree $s=0, 1, 2$. Putting $k=\sqrt{(fMm)}\kappa$, we reduce the effective potential $-R_0$ of the body to the form $-R_0=fMW$ where

$$W=G_1(a)+G_1(-a)+G_2(a)+G_2(-a)+\tfrac{1}{2}\kappa^2(r^2+a^2)^{-1}\quad (G_i=F_i\,/\,fM)$$

The constants r and θ in (1.2) correspond to critical points of the effective potential, i.e. critical points of the function W, while the constant $\dot{\varphi}$ in (1.2) is given by relation (1.1).

2. We consider the system of equations

$$\frac{\partial W}{\partial \theta}\equiv\frac{ar}{2}[m_1(Q_1-P_1)\sin\theta+m_2(P_2-Q_2)\cos\theta]=0 \tag{2.1}$$

$$\frac{\partial W}{\partial r}\equiv\frac{m_1}{2}[P_1(r+a\cos\theta)+Q_1(r-a\cos\theta)]+$$

$$+\frac{m_2}{2}[P_2(r+a\sin\theta)+Q_2(r-a\sin\theta)]-\frac{\kappa^2 r}{(r^2+a^2)^2}=0 \tag{2.2}$$

$$P_i=(r^2+a^2+2ar\gamma_i)^{-3/2},\ Q_i=(r^2+a^2-2ar\gamma_i)^{-3/2}\ (i=1,2)$$

Equation (2.1) is satisfied identically by the values $\theta=0$ and $\theta=\pi/2$ (modπ). Equation (2.2) then takes the form

$$\kappa^2=H_{ij}(r),\ H_{ij}(r)=m_i(r^2+a^2)^3(r^2-a^2)^{-2}r^{-1}+m_j(r^2+a^2)^{1/2} \tag{2.3}$$
$$(i=1,\ j=2 \text{ when } \theta=0 \text{ or } i=2,\ j=1 \text{ when } \theta=\pi/2)$$

We note the obvious properties of the function $H_{ij}(r)(r\in(a,\ +\infty))$

$$H_{ij}(r)>0,\ \lim_{r\to\infty}H_{ij}(r)=\lim_{r\to a+0}H_{ij}(r)=+\infty$$

and consider the equation $H'_{ij}(r)=0$ which can be represented in the form

$$\frac{m_j}{m_i}=-\frac{(r^4-10r^2a^2+a^4)(r^2+a^2)^{5/2}}{(r^2-a^2)^3r^3}\equiv\mu(r) \tag{2.4}$$

Henceforth the prime denotes differentiation.

When $r\in(a,\ +\infty)$ the function $\mu(r)$ decreases monotonically from $+\infty$ to $-1+0$ since

$$\lim_{r\to a+0}\mu(r)=+\infty,\ \lim_{r\to+\infty}\mu(r)=-1,\ \mu'(r)<0\ \ \forall r>a$$

and therefore takes all positive values. Thus when $r>a$ Eq. (2.4) has a unique root $r=r_{ij}^0$ for all values of the parameter $m_j/m_i\in(0;\ +\infty)$. The function $H_{ij}(r)$ takes its minimum value

$$H_{ij}(r_{ij}^0)=\kappa_{ij}^{0^2}>0.$$

at the point r_{ij}^0.

Hence Eq. (2.3) has no solutions when $\kappa^2 < \kappa_{ij}^{0^2}$, has a unique solution $r = r_{ij}^0$ when $\kappa^2 = \kappa_{ij}^{0^2}$ and two families of solutions $r = r_{ij}^{\pm}(\kappa^2)$ when $\kappa^2 > \kappa_{ij}^{0^2}$, where

$$r_{ij}^{+}(\kappa^2) > r_{ij}^0 > r_{ij}^{-}(\kappa^2);\ \ H_{ij}^{'} \gtrless 0 \text{ when } r = r_{ij}^{\pm}(\kappa^2)$$

Henceforth the upper inequality sign corresponds to the branch denoted by the plus sign, and the lower sign corresponds to the branch with the minus sign.

We shall assume further without loss of generality that $m_1 > m_2$. Then $r_{12}^0 > r_{21}^0$ because $\mu(r)$ is a monotonically decreasing function (see Eq. (2.4) from which the values r_{12}^0 and r_{21}^0 are obtained, the first when $i=1$, $j=2$ and the second when $i=2$, $j=1$). Moreover, we also have the inequality $H_{12}(r) > H_{21}(r) \forall > a$, because

$$H_{12}(r) - H_{21}(r) = (m_1 - m_2)[(r^2+a^2)^{5/2} - (r^2-a^2)^2 r](r^2+a^2)^{1/2}(r^2-a^2)^{-2} r^{-1} > 0$$

Obviously, solutions of the form

$$\theta = 0,\ \ r = r_{12}^{\pm}(\kappa^2) \text{ и } \theta = \pi/2,\ \ r = r_{21}^{\pm}(\kappa^2) \tag{2.5}$$

correspond to an orientation of the body in which one of its principal central axes of inertia is directed along the radius vector of the centre of mass, and the other along the tangent to the orbit.

3. We elucidate the nature of the critical points (2.5) of the effective potential, to which end we compute the matrix coefficients of the second variation of W

$$\left(\frac{\partial^2 W}{\partial r^2}\right)^{(ij)} = \left(\frac{r}{(r^2+a^2)^2} H_{ij}^{'}\right)^{(ij)},\ \ \left(\frac{\partial^2 W}{\partial r \partial \theta}\right)^{(ij)} = 0$$

Henceforth the expression $(\ldots)^{(ij)}$ means that the function in brackets has been computed for $\theta = 0$, $r = r_{12}^{\pm}(\kappa^2)(i=1,\ j=2)$ or for $\theta = \pi/2$, $r = r_{21}^{\pm}(\kappa^2)(i=1,\ j=2)$.

Thus the sign of $\partial^2 W/\partial r^2$ is the same as the sign of $H_{ij}^{'}$, i.e.

$$\left(\frac{\partial^2 W}{\partial r^2}\right)^{(ij)} \gtrless 0 \text{ when } r = r_{ij}^{\pm}(\kappa^2)$$

In order to clarify the sign of the expression

$$\left(\frac{\partial^2 W}{\partial \theta^2}\right)^{(ij)} = a^2 r\left(m_i \frac{3r^2+a^2}{(r^2-a^2)^3} - m_j \frac{3r}{(r^2+a^2)^{5/2}}\right)^{(ij)}$$

we investigate the equation $\partial^2 W/\partial \theta^2 = 0$ which can be represented in the form

$$\frac{m_j}{m_i} = \frac{(3r^2+a^2)(r^2+a^2)^{5/2}}{3r(r^2-a^2)^3} \equiv \nu(r) \tag{3.1}$$

When $r \in (a;+\infty)$ the function $\nu(r)$ decreases monotonically from $+\infty$ to $1+0$ since

$$\nu(r) > 0,\ \ \nu'(r) < 0,\ \ \lim_{r \to a+0} \nu(r) = +\infty,\ \ \lim_{r \to +\infty} \nu(r) = 1$$

Hence $\nu(r) > 1 \forall r > a$, i.e. Eq. (3.1) has no solutions when $i=1$, $j=2$ (we recall that $m_1 > m_2$), and has the unique solutions $r = r_{21}^*$ when $i=2$, $j=1$.

Thus

$$(\partial^2 W/\partial\theta^2)^{(12)} > 0, \ r > a; \ (\partial^2 W/\partial\theta^2)^{(21)} > 0(<0), \ r < r_{21}^*(> r_{21}^*)$$

We shall ascertain the relative positions of the points r_{21}^0 and r_{21}^*. The first of these is defined by Eq. (2.4) and the second by Eq. (3.1) (with $i=2$, $j=1$ in both cases). When $r \in (a; +\infty)$ the functions $\mu(r)$ and $\nu(r)$ decrease monotonically from $+\infty$ to -1 and $+1$, respectively, and intersect at the unique point

$$r = r_{\mu\nu} = a[(29+\sqrt{769})/12]^{1/2} \cong 2.174a$$

Here

$$\mu(r) > \nu(r), \ (\mu(r) < \nu(r)) \quad \text{for} \quad r < r_{\mu\nu}(r > r_{\mu\nu})$$

Consequently, $r_{21}^0 > r_{21}^*$ if

$$\frac{m_2}{m_1} < \frac{1}{\mu(r_{\mu\nu})} = \frac{1}{\nu(r_{\mu\nu})} = \frac{(29+\sqrt{769})^{1/2}(17+\sqrt{769})^3}{(41+\sqrt{769})^{5/2}(33+\sqrt{769})} \equiv \delta \cong 0,283$$

and $r_{21}^0 < r_{21}^*$ if $m_2/m_1 > \delta$.

Thus when $m_1 > m_2$ the solution $\theta = 0$, $r = r_{12}^+(\kappa^2)$ is stable in the secular sense (the degree of instability $\chi = 0$), and the solution $\theta = 0$, $r = r_{12}^-(\kappa^2)$ is unstable $(\chi = 1)$.

When $m_1 > m_2 > m_1\delta$ the solution $\theta = \pi/2$, $r = r_{21}^*(\kappa^2)$ is stable in the secular sense $(\chi = 0)$ if $r_{21}^+(\kappa^2) < r_{21}^*$, and unstable $(\chi = 1)$, if $r_{21}^+(\kappa^2) > r_{21}^*$, while the solution $\theta = \pi/2$, $r = r_{21}^-(\kappa^2)$ is always unstable $(\chi = 1)$.

When $m_2 < m_1\delta$ the solution $\theta = \pi/2$, $r = r_{21}^*(\kappa^2)$ is always unstable $(\chi = 1)$, while the solution $\theta = \pi/2$, $r = r_{21}^-(\kappa^2)$ is unstable $(\chi = 1)$, if $r_{21}^-(\kappa^2) < r_{21}^*$ and secularly unstable $(\chi = 2)$ if $r_{21}^-(\kappa^2) < r_{21}^*$; in the latter case gyroscopic stabilization is possible.

Finally, when $m_2/m_1 = \delta$ the two Poincaré stability coefficients vanish simultaneously at the point $r = r_{21}^0$, and the solution $\theta = \pi/2$, $r = r_{21}^{\pm}(\kappa^2)$ is always unstable $(\chi = 1)$.

4. At $\theta = \pi/2$, $r = r_{21}^*$, $\kappa^2 = \kappa_{21}^{*2} \equiv H_{21}(r_{21}^*)$ the second derivative with respect to θ of the effective potential vanishes and (when $m_2/m_1 \neq \delta$) the degree of instability of the corresponding solution $\theta = \pi/2$, $r = r_{21}^{\pm}(\kappa^2)$ changes. As a result, solutions of system (2.1), (2.2) for which $\theta = \pi/2 \pm \psi(\kappa^2)(0 < \psi(\kappa^2) < \pi/2)$ branch off from this solution at this point. These solutions correspond to orientations of the body such that neither of the principal central axes of inertia coincide with either the radius vector of the centre of mass or the tangent to the orbit.

We will indicate the main properties of the corresponding steady motions

$$\theta = \pi/2 \pm \psi(\kappa^2), \ r = \rho(\kappa^2) \tag{4.1}$$

First we note that when $\theta \neq 0$, $\pi/2(\mathrm{mod}\,\pi)$ Eqs (2.1) and (2.2) can be represented, respectively, in the form

$$m_2/m_1 = \Phi(r,\theta) \text{ and } \kappa^2 = \Psi(r,\theta)$$

Here

$$\Phi(r,\theta) = \frac{P_1 - Q_1}{P_2 - Q_2}\,\mathrm{tg}\,\theta, \ \Psi(r,\theta) = \frac{(r^2+a^2)^2}{2r}\{m_1[P_1(r + a\cos\theta) +$$

$$+Q_1(r-a\cos\theta)]+m_2[P_2(r+a\sin\theta)+Q_2(r-a\sin\theta)]\}$$

Analysis of the function $\Phi(r,\ \theta)(\theta\in(0,\ \pi/2),\ r\in(a,\ +\infty))$ shows that

$$\Phi(r,\pi/4)\equiv 1,\quad \lim_{r\to+\infty}\Phi(r,\theta)\equiv 1$$

$$\lim_{\theta\to+0}\Phi(r,\theta)=\nu(r),\quad \lim_{\theta\to\pi/2-0}\Phi(r,\theta)=1/\nu(r)$$

$$\Phi(a+0,\theta)>\Phi(r,\theta)>\Phi(+\infty,\theta)\ \text{when}\ \theta\in(0,\pi/4)$$

$$\Phi(a+0,\theta)<\Phi(r,\theta)<\Phi(+\infty,\theta)\ \text{when}\ \theta\in(\pi/4,\pi/2)$$

Hence when $m_2/m_1<1$ the steady motions (4.1) satisfy the following conditions

$$\theta\in(\pi/2-\psi_*,\pi/2+\psi_*),\quad r\in(a,r_{21}^*]$$

Here $\psi_*=\pi/2-\theta_*$ and θ_* is a root of the equation

$$\frac{m_2}{m_1}=\Phi(a+0,\theta)\equiv\frac{[(1+\cos\theta)^{3/2}-(1-\cos\theta)^{3/2}]\cos^2\theta}{[(1+\sin\theta)^{3/2}-(1-\sin\theta)^{3/2}]\sin^2\theta}\equiv\Phi_*(\theta)$$

Since the relation

$$\lim_{\theta\to+0}\Phi_*(\theta)=+\infty,\quad \Phi_*(\pi/4)=1,\quad \Phi_*(\pi/2)=0,\quad \Phi_*(\theta)<0$$

holds when $\theta\in(0,\ \pi/2)$, this root lies in the range $(\pi/4;\pi/2)$, with θ_* tending to $\pi/4+0$, if m_2/m_1 tends to $1-0$. Hence the deviation of θ from $\pi/2$ does not exceed $\psi_*\in(0;\pi/4)$ and asymptotically approaches the limiting deviation ψ_* when $r\to a+0$. Here (see the equation $\kappa^2=\Psi(r,\ \theta)$) κ^2 tends to the value

$$\kappa_*^2=2a\left\{m_1\frac{(1+\cos\theta_*)^{1/2}+(1-\cos\theta_*)^{1/2}}{\sin\theta_*}+m_2\frac{(1+\sin\theta_*)^{1/2}+(1-\sin\theta_*)^{1/2}}{\cos\theta_*}\right\}$$

Depending on the mass ratio m_2/m_1, the value of κ_*^2 can be smaller or greater than κ_{21}^{*2}. In particular, if m_2/m_1 is near to unity, then $\kappa_*^2<\kappa_{21}^{*2}$, whereas if $m_2/m_1\ll 1$, then $\kappa_*^2>\kappa_{21}^{*2}$.

5. The steady motions of the body (1.2) define a line L in $(r,\ \theta,\ \kappa^2)$ space given by relations (2.1) and (2.2). Figures 1 and 2 show projections of this line onto the $(r,\ \kappa^2)$ plane. Curves P, Q and R correspond to motions for which $\theta=0,\ \pi/2,\ \pi/2\pm\Psi(\kappa^2)(\text{mod}\,\pi)$. Here curves P and Q are projections of plane branches of the line L corresponding to solutions (2.5), and curve R is the projection of a pair of three-dimensional branches corresponding to solutions (4.1). The numbers (0), (1) and (2) indicate the degree of instability of the corresponding steady motions.

The forms of the curves P and Q and the distributions of the degrees of instability are shown in Figs 1 and 2 for the cases $1>m_2/m_1>\delta$ and $m_2/m_1<\delta$, while the form of the curve R and the distribution of the degrees of instability (according to the general ideas of bifurcation theory [1]) are shown for the cases $1-m_2/m_1\ll 1$ and $m_2/m_1\ll 1$.

The points A_1, A_2 and $B^{\pm}$ are branch points of the line L. At these points system (2.1), (2.2) loses local uniqueness of the solutions (at points A with respect to the variable r, and at points B with respect to the variable θ; in the latter case the pair of branches of the line L corresponding to solutions of (4.1) leaves the $\theta=\pi/2$ plane transversally to the latter).

Note that the bifurcation value r_{21}^* tends to $+\infty$ when $m_2/m_1\to 1-0$. This means that even in those cases when the size of the body is small compared to the radius of the centre of mass orbit, motions can be secularly stable with the major axis of the inertial ellipsoid directed along

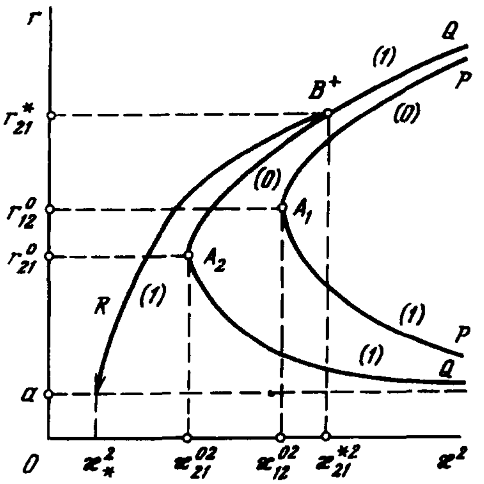

Fig. 1.

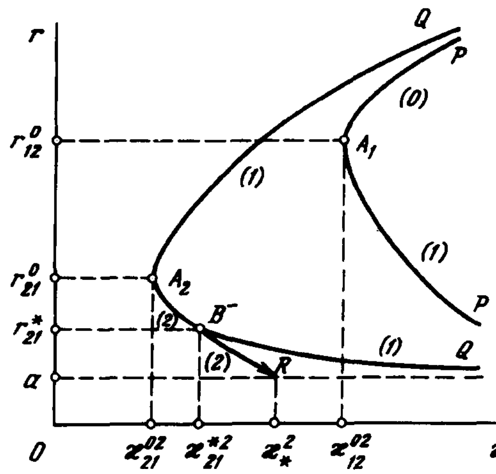

Fig. 2.

the tangent to the orbit, the intermediate axis along the radius vector, and the minor axis (here null) along the normal to the orbital plane. Moreover, steady motions exist for which two of the three principal axes of inertia do not coincide with the axes of the orbital system of coordinates. These results depend on the use of the exact expression for the attractive potential and are impossible in principle when the "satellite" approximation for this potential is used (see also [2–4]).

To conclude, we note that if $m_2/m_1 = 1-\varepsilon$ $(0<\varepsilon\ll 1)$, then $r_{21}^* = \sqrt{(13/2)}a\varepsilon^{-1/2}(1+o(1))$. When $r \sim r_{21}^*$ both the remaining and rejected components in the satellite approximation to the gravitational potential have the same order of smallness

$$(1-m_2/m_1)(a/r)^2 \sim \varepsilon^2, \ (a/r)^4 \sim \varepsilon^2$$

We recall that the satellite approximation to the gravitational potential is obtained from the exact expression for this potential by neglecting all terms of order $(a/r)^3$ and above, where a is the characteristic size of the body and r is the distance from the centre of mass to the centre of attraction, and that terms of order $(a/r)^2$ are retained irrespective of the order of magnitude of the coefficients in front of these terms. In the given problem, when the potential is expanded as a series in powers of a/r, there are no terms of order $(a/r)^3$ terms of order $(a/r)^4$ have coefficients of order unity, while terms of order $(a/r)^2$, corresponding to the orientation of the body have, under the above-stated conditions $m_2/m_1 = 1-\varepsilon$, $r \sim r_{21}^*$, coefficients of order $(a/r)^2$.

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REFERENCES

1. CHETAYEV N. G., *Stability of Motion. Papers on Analytical Mechanics.* Izd. Akad. Nauk SSSR, Moscow, 1962.
2. BELETSKII V. V. and PONOMAREVA O. N., Parametric analysis of the stability of relative equilibrium in a gravitational field. *Kosmich. Issled.* **28**, 5, 664–675, 1990.
3. KARAPETYAN A. V. and SHARAKIN S. A., On the steady motions of two mutually gravitating bodies and their stability. *Vestn. MGU Ser. Mat. Mekh.* **3**, 42–48, 1992.
4. KARAPETYAN A. V., On the bifurcation of the steady motions of two mutually gravitating bodies. In *Investigations of Stability and Stabilization of Motion*, pp. 20–26, Vychisl. Tsentr. Ross. Akad. Nauk, Moscow, 1993.

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